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DYNAMIC PROGRAMMING IN PROBLEMS OF IDENTIFYING DISTRIBUTED PARAMETER SYSTEMS†

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The problem of identifying the input of a system governed by a "semi-linear" evolution equation of parabolic type, based on the results of observations subject to undefined disturbances, is investigated. Estimates of the input, optimal in the sense of the so-called H_∞-criterion, are obtained. The information function of the system—the value function in an appropriate optimization problem—is evaluated. The relations between the information function and information sets are indicated. Optimality principles adequate to the proposed formulations of the problem are formulated and the corresponding dynamic programming equations are derived. Procedures for regularizing the problem, based on evolution equations of the input estimates, are proposed for the heat-conduction equation. © 1999 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Let Ω be a bounded domain in R^n, $n \geqslant 1$; $\partial\Omega = \Gamma \in C^2$. Consider the "semi-linear" evolution equation

$$u_t - \Delta u + f(u) = b(\upsilon(t), t), \quad x \in \Omega, \quad t \in (0, \Theta) \tag{1.1}$$

$$u(\zeta, t) = 0, \quad \zeta \in \Gamma, \quad t \in (0, \Theta]; \quad u(x, 0) = u_0(x), \quad x \in \Omega$$

where Δ is the Laplacian and $\upsilon(t)$ is a "control"; the admissible controls are functions $\upsilon(\cdot) \in L^2(0, \Theta; V)$ such that

$$\upsilon(t) \in \mathbf{V} \text{ for a.e. (almost all) } t \in [0, \Theta],$$

where $\mathbf{V}$ is a closed bounded convex subset of a Hilbert space V. The function $b(\upsilon, t)$, $\upsilon \in V$, $t \in [0, \Theta]$ satisfies a Lipschitz condition as a function of υ and is continuous in t, with $b(0, t) = 0$. As to the function $f(s)$, $s \in R$, the following assumptions are made.

Assumption 1.1. We assume that
1. $f(\cdot) \in C^1(R)$,
2. $f(0) = 0$,
3. f satisfies the growth condition

$$|f(s_1) - f(s_2) - f'(0)(s_1 - s_2)| \leq C(|s_1|^{q-1} + |s_2|^{q-1})|s_1 - s_2|, \quad \forall s_1, s_2 \in R$$

for some $C > 0$ and for q such that $1 < q \leqslant (n + 4)/n$.

It is well known [1–4] that, under the above assumptions, one can prove a local theorem that guarantees the existence of $\gamma > 0$ such that, if $\| u_0 \|_{L^2(\Omega)} + \| \upsilon(\cdot) \|_{L^2(0, \Theta; V)} \leqslant \gamma$, then Eq. (1.1) has a solution, which is moreover unique in the class $C([0, \Theta]; L^2(\Omega)) \cap L^2(0, \Theta; \overset{\circ}{H}{}^1(\Omega))$ and satisfies a Lipschitz condition with respect to u_0, $\upsilon(\cdot)$. An element $u(\cdot, t) \in L^2(\Omega)$ will be called the "state" of system (1.1) at time t.

Let us assume that u_0 and $\upsilon(\cdot)$ are not known in advance but that the system has a solution $u(x, t)$ in the interval $t \in [0, T]$, $T \leqslant \Theta$. It will be assumed that the accessible information about the solution is obtained by virtue of an equation of measurements

$$y(t) = g(u(\cdot, t), t + \xi(t), \quad t \in [0, T] \tag{1.2}$$

where $y(t) \in R^m$ are the measurement data at time t, $\xi(t) \in R^m$ is the measurement noise and $\xi(\cdot) \in L_m^2(0, T)$; $g(\cdot, \cdot) \in C_m(L^2(\Omega) \times [0, T])$ is a given m-dimensional function. We put $z = \{u_0, \upsilon(\cdot), \xi(\cdot)\}$; we

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call an element $z \in Z \equiv L^2(\Omega) \oplus L^2(0, T; V) \oplus L^2_m(0, T)$ an "input" to system (1.1), (1.2) ($\oplus$ denotes the direct sum of Hilbert spaces). Define $\mathbf{Z} \subseteq Z$ to be the set of all inputs in which $\upsilon(t) \in \mathbf{V}$ for almost all t. We will also use the following notation

$$w = \{u_0, v(\cdot)\}, \quad w \in W \equiv L^2(\Omega) \oplus L^2(0, T; V)$$
$$\mathbf{W} = \{w \in W | v(t) \in \mathbf{V} \quad \text{for almost all} \quad t \in [0, T]\}$$

Let $u(x, t; w)$ denote the solution of Eq. (1.1) corresponding to the initial state u_0 and the control $\upsilon(\cdot)$ in the vector $w = \{u_0, \upsilon(\cdot)\}$.

The problem is to estimate the "input" $z \in Z$ based on the "output" $y(\cdot) \in L^2_m(0, T)$. In some applications, this inverse problem is conventionally referred to as "inverting" the system.

2. THE H_∞-APPROACH AND MINIMAX ESTIMATION OF THE INPUT

We define a functional in the input space Z of system (1.1), (1.2) by

$$F(z, T) = \alpha(u_0) + \int_0^T f_0(v(t), \xi(t), t)dt \tag{2.1}$$

The continuous functionals $\alpha(u_0)$, $u_0 \in L^2(\Omega)$ and $f_0(\upsilon, \xi, t)$, $\{\upsilon, \xi, t\} \in V \times R^m \times [0, T]$ are such that $F(\cdot, T)$ is strictly convex and coercive for all $T \in (0, \Theta]$; the latter means that, for some $c_0 = c_0(T) > 0$, we have $F(z, T) \geqslant c_0\|z\|^2_Z$, $\forall Z \in Z$.

Definition 2.1. An H_∞-estimate of the input of system (1.1), (1.2) relative to a criterion F based on observations $y(t)$, $t \in [0, T]$ is an element $\mathbf{z}^* = \mathbf{z}^*(y) \in Z$ such that the quantity $\kappa^2 = \kappa^2(y)$, satisfying the inequality

$$\|z - \mathbf{z}^*\|^2 \le \varkappa^2 F(z, T) \tag{2.2}$$

for all $z \in \mathbf{Z}$ such that

$$y(t) = g(u(\cdot, t; w), t) + \xi(t), \quad t \in [0, T] \tag{2.3}$$

is the least of all possible such elements.

Remark 2.1. The optimality criterion (2.2) is related to problems investigated in what is known as "H_∞-optimal control theory"—a deterministic approach to the investigation of optimum problems in the presence of disturbances (see, for example, [5–8]). The first results in that direction were formulated in terms of Hardy's Banach space H_∞, whence the terminology.

The functional F may be interpreted as a "measure of uncertainty" in system (1.1), (1.2). Its value determines the accuracy $e^2 = \| z - \mathbf{z}^*_0 \|^2$ of an estimate $\mathbf{z}^*$ of the input z based on observations $y(t)$, $t \in [0, T]$, with constant of proportionality κ^2. It is obvious that κ^2 generally depends on the observed signal $y(\cdot)$ and that

$$\varkappa^2(y) = \inf_{z^* \in Z} \sup \frac{\|z - z^*\|^2}{F(z, T)} \tag{2.4}$$

The supremum is evaluated for all non-zero $z \in \mathbf{Z}$ that satisfy Eq. (1.2). If the infimum in (2.4) is achieved, the element at which it is achieved is an H_∞-estimate of the input of system (1.1), (1.2).

Definition 2.2. An H_∞-estimate of the state $U(\cdot, T)$ of system (1.1) relative to as criterion F, based on observations $y(t)$, $t \in [0, T]$, and Eq. (1.2) is an element $\mathbf{u}^* \in L^2(\Omega)$ such that $\kappa^2_u = \kappa^2_u(y)$ is the least possible number satisfying the inequality

$$\|u(\cdot, T) - \mathrm{u}^*\|^2_{L^2(\Omega)} \le \varkappa^2_u F(z, T)$$

for all $z \in \mathbf{Z}$ such that (2.3) holds almost everywhere.

We now consider input estimation within the framework of what is known as the minimax approach, when the undefined noise is subject to *a priori* known restrictions [8–10]. Suppose system (1.1), (1.2) produces an observed signal $y = y(t)$, $t \in [0, T]$, and at the same time it is known that the value of the functional F will never exceed $\mu^2 > 0$.

Definition 2.3. The information set $\mathbf{Z}[T; y] \subseteq Z$, consistent with the observed signal $y(t)$, $t \in [0, T]$, is defined as the set of all elements $z \in \mathbf{Z}$ which satisfy Eq. (1.2) and the condition $F(z, T) \leq \mu^2$.

Clearly, the unknown input z certainly belongs to the set $\mathbf{Z}[T; y]$ and the latter is a guaranteed estimate of the input based on the observation $y(t)$, $t \in [0, T]$, under the constraint $F \leq \mu^2$. Since $\mathbf{Z}[T; y]$ is a bounded set, we can define a minimax estimate of the input as, say, the Chebyshev centre $z^*(T)$ of the set

$$\sup_{z \in \mathbf{Z}[T;y]} \left\| z - z^*(T) \right\| = \min_{z_1 \in Z} \sup_{z \in \mathbf{Z}[T;y]} \left\| z - z_1 \right\|$$

The accuracy of this estimate equals the Chebyshev radius $r^2(T) = \sup \{\|z - z^*(T)\| \mid z \in \mathbf{Z}[T; y]\}$ of the information set. Note that in general $z^*(T) \notin \mathbf{Z}[T; y]$. We also define the information set $\mathbf{U}[T; y] \subseteq L^2(\Omega)$ of states of system (1.1) at time $t = T$, consistent with the observed signal $y(t)$, $t \in [0, T]$, as

$$\mathbf{U}[T; y] = \{u(\cdot) = u(\cdot, T; w) \mid \exists \xi(\cdot) \in L^2_m(0, T) : z \in \mathbf{Z}[T; y]\}$$

The minimax estimate $u^*(T)$ of the state $u(\cdot, T)$ and its accuracy are defined by analogy with the definition for the input.

Consider the following assumption (according to which the system is linear-quadratic).

Assumption 2.1. Assume that system (1.1), (1.2) and criterion (2.1) satisfy the following conditions

1. $\mathbf{V} = V$,
2. $b(v, t) = B(t)v, \quad B(t) \in \mathcal{L}(V, L^2(\Omega)), \quad t \in [0, T]$
3. $f(u) = u$
4. $g(u, t) \equiv G(t)u, \quad G(t) \in \mathcal{L}(L^2(\Omega), R^m), \quad t \in [0, T]$
5. $\alpha(u_0) = \langle u_0 - \bar{u}_0, \ N_1(u_0 - \bar{u}_0)\rangle_{L^2(\Omega)}$
6. $f_0(v, \xi, t) = \langle v - \bar{v}(t), \ N_2(t)(v - \bar{v}(t))\rangle_V + (\xi - \bar{\xi}(t))' M(t)(\xi - \bar{\xi}(t))$

where N_1, $N_2(t)$ and $M(t)$ are self-adjoint positive continuous coercive operators in the respective spaces $L^2(\Omega)$, V, R^m, and $N_2(t)$, $M(t)$ and also $B(t)$ and $G(t)$ are continuous in t in the operator norm. The input $\bar{z} = \{\bar{u}_0, \bar{\upsilon}(\cdot), \bar{z}(\cdot)\} \in Z$ is given.

Henceforth we shall put $Nw \equiv \{N_1 u_0, N_2(\cdot)\upsilon(\cdot)\}$, $My \equiv M(\cdot)y(\cdot)$. We let $\langle \cdot, \cdot \rangle_H$ denote the scalar product in the Hilbert space H. Define operators by

$$C : W \ni w \mapsto g(u(\cdot, \cdot; w), \cdot) \in L^2_m(0, T)$$

$$A : W \ni w \mapsto \{w, y - Cw\} \in W \oplus L^2_m[0, T] \equiv Z$$

Theorem 2.1. Let assumption 1.1 be valid. Then a unique element $\mathbf{z}^* = \{\mathbf{w}^*, \xi^*\} \in Z$ exists which is an H_∞-estimate of the input z relative to the criterion F, based on observations $y(t)$, $t \in [0, T]$, such that, moreover

$$\mathbf{w}^* = PC^*M(y - C\overline{w} - \bar{\xi}) + \overline{w}, \quad \xi^* = y - C(\overline{w} + \mathbf{w}^*) \tag{2.5}$$

$$\varkappa^2 = \| APA^* \|, \quad P = (N + C^*MC)^{-1}$$

Proof. Under our assumptions about the system, the problem reduces to evaluating, for given y, the least number $\kappa^2 = \kappa^2(y)$ such that, for some $z^* \in Z$

$$\| Aw - z^* \|_W^2 \leq \varkappa^2 [\langle u_0 - \bar{u}_0, N_1(u_0 - \bar{u}_0)\rangle_{L^2(\Omega)} +$$

$$+\int_0^T ((y(t) - (Cw)(t) - \bar{\xi}(t))' M(t)(y(t) - (Cw)(t) - \bar{\xi}(t))$$

$$+\langle v(t) - \bar{v}(t), N_2(t)(v(t) - \bar{v}(t))\rangle_V) dt], \ \forall w \in W \tag{2.6}$$

Having prescribed an arbitrary $\mu^2 > 0$, consider the set W_{μ^2} of all elements $w \in W$ such that the coefficient of κ^2 on the right of inequality (2.6) does not exceed μ^2. Let us assume that μ^2 is sufficiently large and that $W_{\mu^2} \neq 0$. Then the supporting function of the set $AW_{\mu^2} = \{z \in Z \mid z = Aw, w \in W_{\mu^2}\}$ is

$$\rho(\lambda \mid AW_{\mu^2}) \equiv \sup_{w \in W_{\mu^2}} \langle A^*\lambda, w\rangle =$$

$$= \langle \lambda, A\mathbf{w}^*\rangle + (\mu^2 - \varkappa^2(y))^{1/2} \langle \lambda, \ APA^*\lambda\rangle_Z^{1/2}, \ \lambda \in Z \tag{2.7}$$

where $\mathbf{w}^*$ and P are those specified in the assumptions of the theorem. Since

$$\sup_{w \in W_{\mu^2}} \| Aw - A\mathbf{w}^* \|^2 = \| APA^* \| (\mu^2 - \varkappa^2(y))$$

it follows that $\kappa^2 \leqslant \|APA^*\|$.

We will show that this is an exact equality. Suppose that $\kappa^2 = \|APA^*\| - \varepsilon$, $\varepsilon > 0$ and, accordingly, an H_∞-estimate of the input different from that defined in (2.5) is $\bar{z}^*$. Choose μ^2 so that $\mu^2 > 2\varepsilon^{-1}\kappa^2(y) \|APA^*\|$. It follows from (2.7) and the properties of the Chebyshev centre of the set $AW_\mu 2$ that, for any δ, $0 < \delta < \max 1/2\mu^2 \in \|APA^*\|^{-1}, \|APA^*\|(\mu^2 - \kappa^2(y))\}$, a $w \in W_{\mu^2} \backslash W_{\mu^2 - \delta}$, exists such that

$$\| Aw_\delta - \bar{\mathbf{z}}^* \|^2 \geq \| Aw_\delta - Aw^* \|^2 > (\mu^2 - \delta - \varkappa^2(y)) \| APA^* \|$$

Then $\|Aw_\delta - \bar{\mathbf{z}}^*\|^2 > \kappa^2\mu^2 \geqslant \kappa^2 F(Aw_\delta, T)$, contrary to the definition of κ^2. The theorem is proved.

Corollary 2.1. If assumption 2.1 holds, there is a unique element $\mathbf{u}^*$ which is an H_∞-estimate of the state $u(\cdot, T)$ relative to the criterion F, based on observations $y(t)$, $t \in [0, T]$, and moreover $u^* = u(\cdot T; \mathbf{w}^*)$.

Corollary 2.2. If assumption 2.1 holds, the following equalities are true for system (1.1), (1.2): $\mathbf{z}^* = z^*(T)$, $\mathbf{u}^* = u^*(T)$.

Remark 2.2. All the results obtained previously for systems (1.1), (1.2) under assumption (2.1) may be transferred to systems in which the observations are obtained by virtue of the equation

$$y(t) = G(t)u(\cdot, t) + R(t)v(t) + \xi(t), \ t \in (0, T)$$

where $R(t) \in \mathcal{L}(V, R^m)$ is an operator function continuous in t.

3. INFORMATION STATE AND INFORMATION SET

Let us return to system (1.1), (1.2). Suppose the observed signal $y(t)$, $t \in [0, T]$ is known. We define a "value" function

$$\Phi(\hat{u}, \theta) = \inf_{z \in Z} \{F(z, T) \mid u(\cdot, \theta; w) = \hat{u}$$

$$y(t) = g(u(\cdot, t; w), t) + \xi(t), t \in [0, \theta]\} \tag{3.1}$$

The domain of definition of Φ will be defined as the set of all pairs $(\hat{u}, \theta) \in L^2(\Omega) \times [0, T]$ for which an element $w \in \mathbf{W}$ exists such that $u(\cdot, \theta; w) = \hat{u}$.

Definition 3.1 [5, 8]. The function $\Phi(\hat{u}, \theta)$ is called the information state of system (1.1), (1.2) based on observation $y(\cdot)$ relative to the criterion F (2.1).

Let us assume that, with respect to an input z for which the output $y(t)$, $t \in [0, T]$ was obtained, we also know that

$$F(z, T) \leq \mu^2 \tag{3.2}$$

We will construct the information set $\mathbf{U}[T; y]$.

Lemma 3.1. The following equality holds

$$\{\hat{u} \in L^2(\Omega) \mid \Phi(\hat{u}, T) \leq \mu^2\} = \mathbf{U}[T; y]$$

The proof follows directly from the definitions of the information set $\mathbf{U}[T; y]$ and the function $\Phi(\hat{u}, \theta)$.

Note that in a linear-quadratic system the set $\mathbf{U}[T; y]$ is an ellipsoid with supporting function [9]

$$\rho(l \mid \mathbf{U}[T; y]) = \left\langle l, u^*(T)\right\rangle_{L_2(\Omega)} + (\mu^2 - h^2(T))^{1/2} \langle l, P(T)l\rangle^{1/2}_{L_2(\Omega)} \forall l \in L_2(\Omega)$$

where

$$\dot{u}^*(T) - Du(T) = B(T)\,\bar{v}(T) + P(T)G^*(T)M(T)(y(T) - G(T)(u^*(T) + u(\cdot, T; \overline{w})) - \bar{\xi}(T)), u^*(0) = \bar{u}_0 \tag{3.3}$$

$$\begin{aligned} &\dot{P}(T) - DP(T) - P^*(T)D^* = B(T)N_2(T)B^*(T) - \\ &-P(T)G^*(T)M(T)G(T)P(T), \ P(0) = N_1^{-1} \\ &\dot{h}^2(T) = (y(T) - G(T)(u^*(T) + u(\cdot, T; \overline{w})) - \bar{\xi}(T))' M(T)(y(T) - \\ &-G(T)(u^*(T) + u(\cdot, T; \overline{w})) - \bar{\xi}(T)), \ h^2(0) = 0 \end{aligned} \tag{3.4}$$

Let D denote the operator $Du = \Delta u - u$ with domain of definition $\{u \in H^{01}(\Omega), \Delta u \in L^2(\Omega)\}$. The solution of an operator equation for $P(T)$ is understood in the sense of [11, 12].

Consider a pair, $\hat{u}$, θ in the domain of definition of Φ. Under the assumptions adopted in Section 1, assuming also, for example, that the function $f(u) \equiv f(u(x))$ is sequentially weakly closed on $L^2(\Omega)$, the infimum in (3.1) is achieved. Let $\hat{\mathbf{w}} = \{\hat{\mathbf{u}}_0, \hat{\mathbf{v}}(\cdot)\}$ be a minimum point. By the Bellman Optimum Principle [7, 13, 14], the following equality will hold for $\delta > 0$

$$\Phi(\hat{u}, \theta) = \Phi(u(\cdot, \theta - \delta; \hat{\mathbf{w}}), \theta - \delta) + \int_{\theta - \delta}^{\theta} f_0(\hat{\mathbf{v}}(t), y(t) - g(u(\cdot, t; \hat{\mathbf{w}}), t)dt$$

Consequently, if $y(t)$ and $\hat{\mathbf{v}}(t)$ are continuous from the left at θ, then

$$\lim_{\delta \to +0} \delta^{-1}(\Phi(\hat{u}, \theta) - \Phi(u(\cdot, \theta - \delta; \hat{\mathbf{w}}), \theta - \delta)) = f_0(\hat{\mathbf{v}}(\theta), y(\theta) - g(\hat{u}, \theta), \theta)$$

Similarly, for any δ and any $w \in \mathbf{W}$ satisfying the equality $u(\cdot, \theta; w) = \hat{u}$ and the condition that the function $\upsilon(t)$ is left continuous at θ, we have

$$\lim_{\delta \to 0} \delta^{-1}(\Phi(\hat{u}, \theta) - \Phi(u(\cdot, \theta - \delta; w), \theta - \delta)) \leq f_0(v(\theta), y(\theta) - g(\hat{u}, \theta), \theta)$$

We will use the notation

$$d\Phi(\hat{u}, \theta) / dt\,|_{(1.1)w} = \overline{\lim_{\delta \to 0}}\, \delta^{-1}(\Phi(\hat{u}, \theta) - (\Phi(u(\cdot, \theta - \delta; w), \theta - \delta))$$

for $w \in \mathbf{W}$ such that $u(\cdot, \theta; w) = \hat{u}$.

Theorem 3.1. For $\hat{u}$ and θ in the domain of definition of the function Φ for which θ is a left-continuity point of the functions $\hat{\mathbf{v}}(t)$ and $y(t)$ the following equation holds

$$\min_{w}\{f_0(\upsilon(\theta), y(\theta)-g(\hat{u},\theta),\theta)-d\Phi(\hat{u},\theta)/dt\,|_{(1.1)w}\}=$$
$$=f_0(\hat{v}(\theta), y(\theta)-g(\hat{u},\theta),\theta)-d\Phi(\hat{u},\theta)/dt\,|_{(1.1)\hat{w}}=0 \tag{3.5}$$

The infimum is evaluated for all $w \in \mathbf{W}$ such that $u(\cdot, \theta; w) = \hat{u}$ and the function $\upsilon(t)$ is left continuous at θ. Henceforth, $\Phi(\hat{u}, 0) = \alpha(\hat{u})$, $\forall\, \hat{u} \in L^2(\Omega)$.

Equation (3.5) is the direct equation of the Hamilton–Jacobi–Bellman type [5, 7, 13], written in implicit form.

Let us assume that the problem of the H_∞-estimate of the input to system (1.1), (1.2) based on measurements $y(t)$, $t \in [0, T]$, relative to the criterion F (2.1), has a solution, and that the quantity $\kappa^2 = \kappa^2(y)$ has been evaluated. Consider the functional

$$L(z^*, z, \theta) = \| u_0 - u_0^* \|^2_{L^2(\Omega)} + \| \upsilon(\cdot) - \upsilon^*(\cdot) \|^2_{L^2(0,\theta;V)} +$$
$$+ \| \xi(\cdot) - \xi^*(\cdot) \|^2_{L^2_m(0,\theta)} - \varkappa^2(y) F(z,\theta)$$

and define a "value" function

$$J(\hat{u}^*, \hat{u}, \theta) = \inf_{z^* \in Z}\sup_{z \in \mathbf{Z}}\{L(z^*, z, \theta) \mid u(\cdot,\theta;w^*) = \hat{u}^*,\ \ u(\cdot,\theta;w) = \hat{u},$$
$$y(t) = g(u(\cdot,t;w),t) + \xi(t), t \in [0,\theta]\} \tag{3.6}$$

We shall say that $\hat{u}^*, \hat{u} \in L^2(\Omega)$, $\theta \in [0, T]$ belong to the domain of definition of J if $z^* \in Z$, $z \in \mathbf{Z}$ exist satisfying the three equalities on the right of the definition (3.6) of the "value" function.

Let us assume that for all $\hat{u}^*, \hat{u}, \theta$ in the domain of definition of J a saddle point $\hat{\mathbf{z}}^*, \hat{\mathbf{z}}$ of the functional L exists

$$\inf_{z^* \in Z}\{L(z^*,\ \hat{z},\ \theta) \mid u(\cdot,\ \theta;\ w^*) = \hat{u}^*\} = L(\hat{z}^*,\ \hat{z},\ \theta) =$$
$$= \sup_{z \in \mathbf{Z}}\{L(\hat{\mathbf{z}}^*,\ z,\ \theta) \mid u(\cdot,\ \theta;\ w) = \hat{u},\ y(t) = g(u(\cdot,\ t;\ w),\ t) + \xi(t),\ t \in [0,\ \theta]\}$$

Given $z^* \in Z$ such that $u(\cdot, \theta; w^*) = \hat{u}^*$ and $z \in \mathbf{Z}$ such that $u(\cdot, \theta; w^*) = \hat{u}^*$ and relation (2.3) holds, we put

$$dJ(\hat{u}^*,\ \hat{u},\ \theta)/dt\,|_{(1.1)w^*,w} = \overline{\lim_{\delta \to 0}}\,\delta^{-1}(J(\hat{u},\ \hat{u}^*,\ \theta) - J(u(\cdot,\ \theta - \delta;\ w^*),$$
$$u(\cdot,\ \theta - \delta;\ w),\ \theta - \delta))$$

Theorem 3.2. For all $\hat{u}^*, \hat{u}, \theta$ in the domain of definition of the function J, if $\hat{\mathbf{z}}^*, \hat{\mathbf{z}}$ is a saddle point of the functional L and the functions $\hat{\mathbf{v}}(t)$, $\hat{\mathbf{v}}^*(t)$, $y(t)$ are left continuous at the point θ, the following equation holds

$$\min_{z^*}\max_{z}\{-dJ(\hat{u}^*,\ \hat{u},\ \theta)/dt\,|_{(1.1)w^*,w} + \| \upsilon(\theta) - \upsilon^*(\theta) \|^2_V +$$
$$+ \| \xi(\theta) - \xi^*(\theta) \|^2_{R^m} - k^2(y) f_0(\upsilon(\theta),\ y(\theta) - g(\hat{u},\ \theta),\ \theta)\} =$$
$$= -dJ(\hat{u}^*,\ \hat{u},\ \theta))/dt\,|_{(1.1)\hat{\mathbf{w}}^*,\hat{\mathbf{w}}} + \| \hat{\mathbf{v}}(\theta) - \hat{\mathbf{v}}^*(\theta) \|^2_V + \| \hat{\boldsymbol{\xi}}(\theta) - \hat{\boldsymbol{\xi}}^*(\theta) \|^2_{R^m} -$$
$$-\varkappa^2(y) f_0(\hat{\mathrm{v}}(\theta),\ y(\theta) - g(\hat{u},\ \theta),\ \theta)\} = 0$$

The extrema are evaluated over all $z^* \in Z$ and $z \in \mathbf{Z}$ such that $u(\cdot, \theta; w^*) = \hat{u}^*$, $u(\cdot, \theta; w^*) = \hat{u}$ and relation (2.3) holds, and the functions $\upsilon^*(t)$, $\upsilon(t)$, $\xi^*(t)$, $\xi(t)$ are left continuous at θ. Furthermore

$$J(\hat{u}^*,\ \hat{u},\ 0) = \left\| \hat{u}^* - \hat{u} \right\|^2_{L^2(\Omega)} - \varkappa^2(y)\alpha(\hat{u}),\ \ \forall \hat{u}^*,\ \hat{u} \in L^2(\Omega)$$

Proof. For any $\delta \in (0, \theta)$, the following equation holds

$$J(\hat{u}^*,\ \hat{u},\ 0)=\min_{z^*\in Z}\ \max_{z\in \mathbf{Z}}\{L(z^*,\ z,\ \theta)\,|\,u(\cdot,\ \theta-\delta;\ w^*)=u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}^*)$$

$$u(\cdot,\ \theta-\delta;\ w)=u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}),\ v^*(t)=\hat{\mathrm{v}}^*(t),\ v(t)=\hat{\mathrm{v}}(t)$$

$$\xi^*(t)=\hat{\xi}^*(t),\ \xi(t)=\hat{\xi}(t),\ t\in(\theta-\delta,\ \theta]$$

$$y(t)=g(u(\cdot,\ t;\ w),\ t)+\xi(t),\ t\in[0,\ \theta-\delta]\}=$$

$$=J(u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}^*),\ \ u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}),\ \theta+\delta)+\left\|\hat{\mathrm{v}}(\cdot)-v^*(\cdot)\right\|^2_{L^2(\theta-\delta,\ \theta;\ V)}+$$

$$+\left\|\hat{\xi}(\cdot)-\xi^*(\cdot)\right\|^2_{L^2_m(\theta-\delta,\ \theta)}-k^2\int_{\theta-\delta}^{\theta}f_0(\hat{\mathrm{v}}(t),\ y(t)-g(u(\cdot,\ t;\ \hat{\mathbf{w}}),\ t),\ t)dt$$

whence it follows (subject to suitable continuity assumptions) that

$$\lim_{\delta\to+0}\delta^{-1}(J(\hat{u},\ \hat{u}^*,\ \theta)-J(u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}^*),\ \ u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}),\ \theta-\delta))=$$

$$=\left\|\hat{\mathrm{v}}(\theta)-\hat{\mathrm{v}}^*(\theta)\right\|^2_V+\left\|\hat{\xi}(\theta)-\hat{\xi}^*(\theta)\right\|^2_{R^m}-k^2(y)f_0(\hat{\mathrm{v}}(\theta),\ y(\theta)-g(\hat{u},\ \theta),\ \theta)$$

On the other hand, for any $\upsilon^*(\cdot)\in L^2(\theta-\delta,\theta;V)$, $\xi^*(\cdot)\in L^2_m(\theta-\delta,\theta)$ and any $z\in\mathbf{Z}$ such that $u(\cdot,T;w)=\hat{u}$ and $y(t)=g(u(\cdot,t;w),t)+\xi(t)$, $t\in[0,\theta]$, we have the following inequalities

$$J(\hat{u}^*,\ \hat{u},\ \theta)\leqslant J(u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}^*),\ u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}),\ \theta-\delta)+\int_{\theta-\delta}^{\theta}(\|\hat{\mathrm{v}}(t)-$$

$$-v^*(t)\|^2_V+\|\hat{\xi}(t)-\xi^*(t)\|^2_{R^m}-k^2(y)f_0(\hat{\mathrm{v}}(t),y(t)-g(u(\cdot,\ t;\ \hat{\mathbf{w}}),\ t),\ t))dt$$

$$J(\hat{u}^*,\ \hat{u},\ \theta)\geqslant J(u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}^*),\ u(\cdot,\ \theta-\delta;\ \hat{\mathbf{w}}),\ \theta-\delta)+\int_{\theta-\delta}^{\theta}(\|v(t)-$$

$$-\hat{\mathrm{v}}^*(t)\|^2_V+\|\xi(t)-\hat{\xi}^*(t)\|^2_{R^m}-k^2(y)f_0(v(t),\ y(t)-g(u(\cdot,\ t;\ w),\ t),\ t))dt$$

The theorem is proved.

4. RETROGRADE EQUATION OF HAMILTON-JACOBI-BELLMAN TYPE

In this section and the next two we will be studying system (1.1), (1.2) under Assumptions 2.1, also assuming, for simplicity, that $\bar{z}=0$. An H_∞-estimate of the input to this system exists and is unique and can be obtained from the condition

$$\mathbf{z}^*=\arg\min_{z^*\in Z}\ \max_{z=\{w,\ \xi\}\in Z}\{\|z-z^*\|^2-\varkappa^2(y)F(z,\ T)\,|$$

$$y(t)=G(t)u(\cdot,\ t;\ w)+\xi(t),\ t\in[0,\ T]\}$$

With the quantity $\kappa^2(y)$ found in Theorem 2.1, having evaluated the maximum over all $z\in\mathbf{Z}$, we obtain the following equation for the projection of the input estimate onto W

$$\mathbf{w}^*=\arg\min_{v^*(\cdot)\in L^2(0,\ T;\ V)}\ \min_{u_0^*\in L^2(\Omega)}\Big\{\left\langle u_0^*,\ N_1u_0^*\right\rangle_{L^2(\Omega)}+\left\langle v^*(t),\ N_2(t)v^*(t)\right\rangle_V+$$

$$+(y(t)-G(t)u(\cdot,\ t;\ w^*))'M(t)(y(t)-G(t)u(\cdot,\ t;\ w^*))\Big\}$$

(Note that $\mathbf{w}^*$ is also an H_∞-estimate for w based on observations $y(t)$, $t\in[0,T]$ relative to the criterion F; this concept is defined by analogy with the definition of an H_∞-estimate for the input.) The minimum with respect to u_0^* is achieved at an element $\mathbf{u}_0^*$, which is a solution of the functional equation

$$N_1\mathbf{u}_0^* + \int_0^T S^*(t)G^*(t)M(t)G(t)S(t)dt\mathbf{u}_0^* =$$

$$= \int_0^T S^*(t)G^*(t)M(t)\,(y(t) - G(t)\int_0^t S(t-\tau)B(\tau)v^*(\tau)d\tau)dt \tag{4.1}$$

(this solution exists and is unique in $L^2(\Omega)$), where $S(t)$, $t \geq 0$, is a strongly continuous semigroup, generated by an operator D. Then

$$\mathbf{v}^*(\cdot) = \arg\min_{v^*(\cdot)}\left\{\int_0^T ((\langle v^*(t),\ N_2(t)v^*(t)\rangle_V + (y(t) -\right.$$

$$\left. -G(t)\check{u}(t))'M(t)\,(y(t) - G(t)\check{u}(t)))dt \mid v^*(\cdot) \in L^2(0,\ T;\ V)\right\} \tag{4.2}$$

where $\check{u}(t)$, $0 \leq t \leq T$ is determined from the equations

$$\dot{\check{u}}(t) - D\check{u}(t) = B(t)v^*(t) + \check{P}(t)G^*(t)M(t)\,(y(t) - G(t)\check{u}(t)) \tag{4.3}$$

$$\check{u}(0) = 0$$

$$\dot{\check{P}}(t) - D\check{P}(t) - \check{P}^*(t)D^* = -\check{P}(t)G^*(t)M(t)G(t)\check{P}(t) \tag{4.4}$$

$$\check{P}(0) = N_1^{-1}$$

Specifying arbitrary $\theta \in [0, T]$, $\check{u}_0 \in L^2(\Omega)$, $\upsilon^*(\cdot) \in L^2(\theta, T; V)$, let us consider a trajectory of system (4.3) over the interval $\theta \leq t \leq T$ with initial data $\check{u}(\theta) = \check{u}_0$. Denote this trajectory by $\check{u}(t, \theta, \check{u}; \upsilon^*)$. Note that, as before, the function $\check{P}(t)$ is governed by Eq. (4.4).

We define a "value" function by

$$\Psi(\check{u}_0,\ \theta) = \min\left\{\int_\theta^T ((\langle v^*(t),\ N_2(t)v^*(t)\rangle_V + (y(t) -\right.$$

$$-G(t)\check{u}(t,\theta,\ \check{u}_0;\ v^*))'M(t)\,(y(t) - G(t)\check{u}(t,\theta,\ \check{u}_0;\ v^*)))dt \mid$$

$$\left. v^*(\cdot) \in L^2(\theta,\ T;\ V)\right\},\quad \theta \in [0,\ T],\quad \check{u}_0 \in L^2(\Omega) \tag{4.5}$$

The function Φ is continuous on $L^2(\Omega) \times [0, T]$, in which case the minimum on the right of (4.5) is achieved, in fact at a unique point, for any values of the arguments $\check{u}_0$, θ. We will use the following notation

$$d_+\Psi(\check{u}_0,\ \theta)/dt\,|_{(4.3),\ v^*} = \lim_{\delta\to 0}\delta^{-1}(\Psi(\check{u}(\theta+\delta,\ \theta,\ \check{u}_0;\ v^*),\ \theta+\delta) - \Psi(\check{u}_0,\ \theta))$$

It can be shown that $\Psi(\check{u}_0, \theta)$ is a solution of a retrograde equation of the Hamilton–Jacobi–Bellman type

$$d_+\Psi(\check{u}_0,\ \theta)/dt\,|_{(4.3),\ \mathbf{v}^*(t)} + \langle \mathbf{v}^*(t),\ N_2(t)\mathbf{v}^*(t)\rangle_V =$$

$$= \min_{v^*\in V}\left\{d_+\Psi(\check{u}_0,\ \theta)/dt\,|_{(4.3),\ v^*(t)} + \langle v^*(t),\ N_2 v^*(t)\rangle_V\right\} =$$

$$= -(y(\theta) - G(\theta)\check{u}_0)'M(\theta)\,(y(\theta) - G(\theta)\check{u}_0) \tag{4.6}$$

and it satisfies the boundary condition $\Psi(\check{u}_0, T) = 0$, $\forall\check{u}_0 \in L^2(\Omega)$. Let $\mathbf{v}^*(t)$, $\theta \leq t \leq T$ achieve the minimum in (4.5). This function is a solution of the following Fredholm equation of the second kind with symmetric non-negative kernel

$$N_2(t)\mathbf{v}^*(t) + \int_\theta^T K(t,\ \tau)\mathbf{v}^*(\tau)d\tau = g(t),\quad \theta \leq t \leq T \tag{4.7}$$

where

$$K(t,\ \tau)=\int\limits_{\max\{t,\ \tau\}}^{T} B^*(t)C^*(s,\ t)G^*(s)M(s)G(s)C(s,\ \tau)B(\tau)ds$$

$$g(t)=\int\limits_{t}^{T} B^*(t)C^*(t,\ \tau)G^*(\tau)M(\tau)(y(\tau)-$$

$$-G(\tau)\int\limits_{\theta}^{\tau} C(\tau,\ s)\check{P}(s)G^*(s)M(s)y(s)ds)d\tau,\quad \theta\leq t,\ \tau\leq T$$

$C(s, t)$ is an almost strong evolution operator corresponding to the operator $(D - \check{P}(t)G^*(t)M(t)G(t))$, $t \geq 0$ [11]. This leads to the following representation for the function Φ

$$\Psi(\check{u}_0,\ \theta)=\int\limits_{\theta}^{T}(y(t)-G(t)u^0(t))'M(t)\,(y(t)-G(t)u^0(t))dt$$

The function $u^0(t)$, $\theta \leq t \leq T$ satisfies a system of differential equations of the form

$$\dot{u}^0(t)-Du^0(t)=P^0(t)G^*(t)M(t)\,(y(t)-G(t)u^0(t)),\ \ u^0(\theta)=\check{u}_0,$$
$$\dot{P}^0(t)-DP^0(t)-P^{0*}(t)D^*=-P^{0*}(t)G^*(t)M(t)G(t)P^0(t)+$$
$$+B(t)N_2(t)B^*(t),\ \ P^0(\theta)=\check{P}(\theta)$$

Let

$$\partial_+\Psi(\check{u}_0,\ \theta)/\partial\theta=\lim_{\delta\to+0}\delta^{-1}(\Psi(\check{u}_0,\ \theta+\delta)-\Psi(\check{u}_0,\ \theta)) \tag{4.8}$$

$$\partial\Psi(\check{u}_0,\ \theta)/\partial\check{u}_0=\Lambda(\check{u}_0,\ \theta)\in(L^2(\Omega))^* \tag{4.9}$$

(the last expression is the Fréchet derivative of the function $\Psi(\check{u}_0, \theta)$ with respect to $\check{u}_0$). We shall identify $\Lambda(\check{u}_0, \theta)$ with an element $L^2(\Omega)$.

Lemma 4.1. If $\check{u}_0$ is in the domain of definition of the operator D and $\theta \in [0, T)$, function (4.8) exists and is continuous. If $\check{u}_0 \in L^2(\Omega)$, $\theta \in [0, T]$, function (4.9) exists and is continuous.

Theorem 4.1. Suppose Assumption 2.1 holds for system (1.1), (1.2). Then, if $\mathbf{w}^* = \{\mathbf{u}_0^*, \mathbf{v}^*(\cdot)\}$ is an H_∞-estimate of the input based on measurements $y(t)$, $0 \leq t \leq T$, and $\check{\mathbf{u}}(t)$ is a solution of Eq. (4.3) for $\upsilon^*(\cdot) = \mathbf{v}^*(\cdot)$, then for almost all $\theta \in [0, T]$ one has a retrograde dynamic programming equation

$$\partial_+\Psi(\check{\mathbf{u}}(\theta),\theta)/\partial\theta+\langle\mathbf{v}^*(\theta),N_2(\theta)\mathbf{v}^*(\theta)\rangle_V+$$
$$+(y(\theta)-G(\theta)\check{\mathbf{u}}(\theta))'M(\theta)\,(y(\theta)-G(\theta)\check{\mathbf{u}}(\theta))+\langle D\check{\mathbf{u}}(\theta)+B(\theta)\mathbf{v}^*(\theta)+$$
$$+\check{P}(\theta)G^*(\theta)M(\theta)\,(y(\theta)-G\,(\theta)\check{\mathbf{u}}(\theta)),\partial\Psi(\check{\mathbf{u}}(\theta),\theta)/\partial\check{u}_0\rangle_{L^2(\Omega)}=0$$

5. ADAPTIVE ESTIMATION OF THE INPUT

Let us assume that an estimate $\mathbf{z}^* = \{\mathbf{u}_0^*, \mathbf{v}^*(\cdot), \xi^*(\cdot)\}$ has been calculated for system (1.1), (1.2), based on observations $y(t)$, $0 \leq t \leq T$. Henceforth, to stress the dependence of the estimate on the observation interval $[0, T]$, we will write $\mathbf{z}^*(T) = \{\mathbf{u}_0^*(T), \mathbf{v}^*(\cdot\,; T), \xi^*(\cdot\,; T)\}$. It is obvious that, generally speaking, $\mathbf{v}^*(t; T) \neq \mathbf{v}^*(t; T + \delta)$, $\xi^*(t; T) \neq \xi^*(t; T + \delta)$ for $0 < t \leq T$, $\delta > 0$, just as $\mathbf{u}_0^*(T) \neq \mathbf{u}_0^*(T + \delta)$. Hence, in order to evaluate an estimate $\mathbf{z}^*(T + \delta)$ based on the previously derived equations, it is not enough to know $\mathbf{z}^*(T)$ and $y(t)$, $T \leq t \leq T + \delta$. In what follows we propose procedures for the approximate calculation of an estimate $\mathbf{z}^*(T + \delta)$ using only observations $y(t)$, $T \leq t \leq T + \delta$ and the estimate $\mathbf{z}^*(T)$ or an approximation thereof. In addition to Assumption 2.1, we take $V = R^p$.

By the results of Section 4, $\mathbf{v}^*(t, T)$ is a solution of Eq. (4.7). Having fixed $t \geq 0$, we differentiate both sides of this equality with respect to T in the interval $T > t$. Letting $Q(t; T) \in \mathscr{L}(L^2(\Omega))$, $0 \leq t \leq T$ denote a solution of the operator equation

$$N_2(t)Q(t;\ T)+\int\limits_0^T\int\limits_{\max\{t,\ \tau\}}^T B^*(t)C^*(s,\ t)G^*(s)M(s)G(s)C(s,\ \tau)\times$$
$$\times B(\tau)dsQ(\tau;\ T)d\tau=B^*(t)C^*(T,\ t)$$

(which indeed has a unique solution [15]), we find the following equation for $\mathbf{v}^*(t, T)$, using the function $\mathbf{u}^*(t)$ introduced above in (3.3)

$$\partial \mathbf{v}^*(t;\ T)/\partial T=Q(t;\ T)G^*(T)M(T)(y(T)-G(T)u^*(T)),\ \ T>t$$
$$\mathbf{v}^*(t;\ t)=0$$

Similarly, we derive the equation

$$\partial Q(t;\ T)/\partial T=Q(t;\ T)(D-\check{P}(T)G^*(T)M(T)G(T))^*-$$
$$-Q(t;\ T)G^*(T)M(T)G(T)(P(T)-\check{P}(T)),\ \ T>t$$
$$Q(t;\ t)=N_2^{-1}(t)B^*(t)$$

These differential equations are the basis of the following procedure for the approximate calculation of an estimate for the control. Fix $\delta>0$ and construct a uniform partition of the interval $[0, T]$, of diameter $\delta>0$ (to fix our ideas, we assume that $T=\delta K$): $t_0=0, t_1=\delta, t_2=t_1+\delta\ldots, t_K=T$. Defining $\mathbf{v}_0^\delta=0$, we proceed successively to calculate $\mathbf{v}_{k+1}^\delta$ for $0\leqslant k\leqslant K-1$, as the values at $t=T$ of the solution of the Cauchy problem

$$\dot{\upsilon}(t)=Q(t_k;\ t)G^*(t)M(t)(y(t)-G(t)u^*(t)),\ \ t_k\leqslant t\leqslant T,\ \ \upsilon(t_k)=0 \tag{5.1}$$

Construct a piecewise-constant function $\mathbf{v}^\delta(t)$, $0\leqslant t\leqslant T$, putting

$$\mathbf{v}^\delta(t)=\mathbf{v}_{k+1}^\delta,\ \ t_k<t\leqslant t_{k+1} \tag{5.2}$$

Theorem 5.1. The functions $\mathbf{v}^\delta(\cdot)$ converge to $\mathbf{v}^*(\cdot;\ T)$ as $\delta\to 0$, in the norm of the space $L_p^2(0, T)$.

Remark 5.1. The values of the function $\mathbf{v}^\delta(t)$ in the interval $T\leqslant t\leqslant T+\delta$ may be found by solving differential equation (5.1) over the interval $T\leqslant t\leqslant T+\delta$ with initial condition $\upsilon(T)=0$, or by using a suitable finite-dimensional approximation

$$\mathbf{v}_{K+1}^\delta=\delta N_2^{-1}(t_K)B^*(t_K)G^*(t_K)M(t_K)(\delta^{-1}\int\limits_T^{T+\delta}y(\tau)d\tau-G(t_K)u^*(t_K))$$

It should be emphasized that the evaluation of the function $u^*(t)$, $t\in(T, T+\delta]$ (with the value of $u^*(T)$ determined on the basis of $y(t)$, $t\in[0, T]$) requires only a knowledge of the observations $y(t)$, $t\in(T, T+\delta]$.

We now consider the estimate $\mathbf{u}_0^*(T)$. This function, as shown previously, is a solution of the functional equation (4.1). We will represent $\mathbf{u}_0^*(T)$ in the form

$$\mathrm{u}_0^*(T)=\tilde{P}(T)\int\limits_0^T\tilde{G}^*(t)M(t)(y(t)-G(t)\int\limits_0^t S(t-\tau)B(\tau)\upsilon^*(\tau;\ T)d\tau)dt \tag{5.3}$$

$$(\tilde{G}(t)=G(t)S(t),\ \ \tilde{P}(t)=(N_1+\int\limits_0^t\tilde{G}^*(\tau)M(\tau)\tilde{G}(\tau)d\tau)^{-1},\ \ t\geqslant 0)$$

Prescribing an arbitrary $\delta>0$, we find $\mathbf{v}^\delta(t)$, $0\leqslant t\leqslant T$. Correspondingly, we let $\mathbf{u}_0^{*\delta}(T)$ denote the expression obtained from (5.3) by replacing $\upsilon^*(\tau;\ T)$ by $\mathbf{v}^\delta(\tau)$. Differentiating with respect to T in the domain $T>0$, we find that

$$d\mathbf{u}_0^{*\delta}(T)/dT=\tilde{P}(T)\tilde{G}^*(T)M(T)(y(T)-\tilde{G}(T)\mathbf{u}_0^{*\delta}(T)-G(T)V(T))$$

$$\mathbf{u}_0^{*\delta}(0)=0$$

$$d\tilde{P}(T)/dT=-\tilde{P}^*(T)\tilde{G}^*(T)M(T)\tilde{G}(T)\tilde{P}(T),\quad \tilde{P}(0)=N_1^{-1}$$

$$dV(T)/dT-DV(T)=B(T)\mathbf{v}^\delta(t),\quad V(0)=0$$

Theorem 5.2. For any $T>0$, the functions $\mathbf{u}_0{}^{*\delta}(T)$ converge as $\delta\to 0$ to $\mathbf{u}_0{}^*(T)$ in the norm of $L^2(\Omega)$.

Remark 5.2. For stationary observation operators $G(\cdot)$, i.e. $G(t)\equiv G$, $0\leqslant t$ [9, 10, 17], the operator $\tilde{G}(t)$ admits of an effective computational form. Consider, for example, spatially averaged observations

$$Gu(\cdot,\ t)=\mathrm{col}\,[\int_\Omega g_1(x)u(x,\ t)dx,\ ...,\ \int_\Omega g_m(x)u(x,\ t)dx] \tag{5.4}$$

where $g_1=g_1(x),\ldots,g_m=g_m(x)$ are given functions which are square-integrable over Ω. Then

$$\tilde{G}u(\cdot,\ t)=\mathrm{col}\,[\int_\Omega \tilde{g}_1(x,\ t)u(x,\ t)dx,\ ...,\ \int_\Omega \tilde{g}_m(x,\ t)u(x,\ t)dx]$$

where $\tilde{g}_1(t)=\tilde{g}_1(x,t),\ldots,\tilde{g}_m(t)=\tilde{g}_m(x,t)$ are solutions of the Cauchy problems

$$\dot{\tilde{g}}_i(t)-D\tilde{g}_i(t)=0,\quad t>0,\quad \tilde{g}_i(0)=g_i,\quad i=1,\ ...,\ m$$

6. EXAMPLE

In addition to Assumption 2.1, we assume the following conditions.

Assumption 6.1. Suppose that
1. the initial state u_0 of system (1.1) is known,
2. $V=R^m$,
3. $B(t)\equiv B$, $t\in[0,T]$, where

$$Bv=b_1(x)v_1+\ldots+b_m(x)v_m,\quad v\in V,\quad x\in\Omega$$

$$b_i\in\overset{\circ}{H}{}^2(\Omega),\quad i=1,\ ...,\ m,$$

4. the observation operator is such that $G(t)\equiv G$, where G is of the form (5.4), and $g_k\in H^1(\Omega)$, $k=1,\ldots,m$,
5. the observation noise $\xi(t)$ in Eq. (1.1) is an element of the space $H_m^1(0,T)$.

With these assumptions, we obtain an equation for $\upsilon(\cdot)$

$$Y(t)-GDu(\cdot,\ t;\ v(\cdot))=GBv(t)+\eta(t),\quad 0\leqslant t\leqslant T \tag{6.1}$$

$$(Y(t)=\dot{y}(t),\quad \eta(t)=\dot{\xi}(t))$$

Here D is the operator introduced in Section 3. By Remark 2.2, we can now determine an H_∞-estimate for the control as a function $\mathbf{v}^*(\cdot)\in L_m^2(0,T)$ such that the quantity κ^2, guaranteeing the inequality

$$\|v(\cdot)-\mathbf{v}^*(\cdot)\|^2_{L^2_m(0,\ T)}\leqslant k^2\int_0^T(v(t)'N(t)v(t)+\eta(t)'M(t)\eta(t))dt$$

for all $\upsilon(\cdot)$, $\eta(\cdot)\in L_m^2(0,T)$ satisfying (6.1), is the least possible. This function, as is easily shown, minimizes the functional

$$J(v^*(\cdot))=\int_0^T(v^*(t)'N(t)v^*(t)+(Y(t)-GDu(\cdot,\ t;\ v^*)-$$

$$-GBv^*(t))'M(t)(Y(t)-GDu(\cdot,\ t;\ v^*)-GBv^*(t)))dt,\quad v^*(\cdot)\in L_m^2(0,\ T)$$

Put $M(t)=E$, $N(t)=\varepsilon E$, where $\varepsilon>0$ and E is the $m\times m$ identity matrix. We put $\mathbf{v}^*(\cdot)=\mathbf{v}_\varepsilon^*(\cdot)$ to emphasize the dependence of the estimate on the choice of the matrices $M(t)$ and $N(t)$.

Theorem 6.1. If the operator GB is invertible on R^m, the limit lim $v^*_\varepsilon(\cdot) = \hat{v}^*(\cdot)$ as $\varepsilon \to +0$ exists. This function $\hat{v}^*(\cdot)$ is the unique solution of the equation

$$Y(t) - GDu(\cdot,\ t;.\hat{v}^*) = GB\hat{v}^*(t), \quad t > 0$$

Note that it is precisely this function $\hat{v}^*(\cdot)$ that was considered in [17, 19] as an estimate of the control, while the quantities $v^*(t; T + \delta)$, $T \leq t \leq T + \delta$ were determined there from the equation $Y(t) - GDu(\cdot, t; v^*) = GBv^*(t)$, $T \leq t \leq T + \delta$, where the initial condition for $u(x, T)$ was $u(\cdot, T) = u(\cdot, T; v^*(\cdot; T))$. This procedure is analogous to that described in Remark 5.1 if one replaces $u^*(t_K)$ in the latter by $u(\cdot, T; v^\delta)$.

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